hep-th/9704081

Geon Statistics and UIR's of the Mapping Class Group *

Rafael D. Sorkin

Instituto de Ciencias Nucleares, UNAM, A. Postal 70-543, D.F. 04510, Mexico
internet address: sorkin@nuclecu.unam.mx
and
Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.

Sumati Surya

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.
internet address: ssurya@suhep.syr.edu

Abstract

Quantum Gravity admits topological excitations of microscopic scale which can manifest themselves as particles — topological geons. Nontrivial spatial topology also brings into the theory free parameters analogous to the θ-angle of QCD. We show that these parameters can be interpreted in terms of geon properties. We also find that, for certain values of the parameters, the geons exhibit new patterns of particle identity together with new types of statistics. Geon indistinguishability in such a case is expressed by a *proper subgroup* of the permutation group and geon statistics by a (possibly projective) representation of the subgroup.

This talk attempts to answer two questions concerning the effect of topology in generally covariant theories: "how many free parameters are there in quantum gravity?" and "do topological geons really act like particles?" By way of comparison, consider the standard model, which contains both continuous parameters (like the masses of the

* To appear in Juan Carlos D'Olivo, Martin Klein, Hector Mendez (eds.), Proceedings of the VII Mexican School of Particles and Fields I Latin American Symposium on High Energy Physics Conference, held November, 1996, Mérida, México (American Institute of Physics).

quarks, the strong coupling constant, and θ_{QCD}) and discrete parameters (like the handedness of the neutrinos, taken relative to θ_{QCD}, say). In that flat space theory, all of the many parameters could be interpreted in terms of the properties of the particles the theory describes, though in some cases the interpretation would be rather indirect. The question here is what additional parameters arise from *spatial topology*, and can they be interpreted in terms of the properties of the topological particles (geons) that quantum gravity describes? To further focus the question for this talk, we will concentrate on the question of particle *statistics*.

To our two questions we will encounter the following answers. There exist in fact very many additional parameters, both continuous and discrete. And the topological excitations show themselves consistently as particle-like, in the sense that all of the new parameters can be interpreted as telling us about one of the following:

- internal geon "qualities" or "quantum numbers"
- geon collision parameters
- statistics of identical geons

This positive outcome may be taken to bolster the interpretation of the topological excitations as particles (geons).

In addition we will see that geons can manifest new types of indistinguishability and statistics, beyond the (very familiar) bosonic and fermionic types and the (somewhat less familiar) para-statistical types. These novel statistics include:

- A statistics based on cyclic subgroups of the permutation group $\mathcal{S}_N$
- Possibly a new statistics based on *projective* representations of $\mathcal{S}_N$ or its subgroups.

However, all of these results assume that the topology is "frozen", in the sense that they presuppose a spacetime of the product form ${}^3M \times \mathbb{R}$. A key question then is ¿which quantum sectors will survive when topology changing processes are incorporated into the theory?. (Here, we mean by sector a specific set of values of the parameters. Each such sector is "superselected" in the approximation that topology change is ignored, but it can be expected to communicate with other sectors in a more general setting.)

[Primes and geons]

Before we begin the formal analysis, let us recall the definition of a geon, and the mathematical decomposition theorem on which it is based. (For more about this theorem, and about the subject of this talk in general, see the more complete exposition in [1], as well as the references therein.) According to this theorem, an arbitrary 3-manifold (without boundary) which is asymptotically $\mathbb{R}^3$ can be expressed as

$$^3M = \mathbb{R}^3 \# P_1 \# P_2 \# \cdots \# P_N, \tag{1}$$

where the P_i are *primes* — manifolds that cannot be built up by "sticking together" smaller manifolds. In fact we will assume further that the primes are all "irreducible", that is, we will exclude the orientable and nonorientable handles (or "worm-holes") from consideration, as they do not seem to be particle-like in the same sense as other primes. By a *geon* then, we mean a "quantized prime" regarded as a particle, or in other words, that object in the quantum theory to which the irreducible prime submanifold corresponds.

To list all possible geons is unfortunately impossible, because there exist an infinite number of primes, not all of which are known. Nevertheless, it is not difficult to visualize an arbitrary geon in a general sort of way: it is the result of excising a polyhedron from $\mathbb{R}^3$, and then performing appropriate identifications on the boundary faces created by the excision. One knows that every prime can be made in this manner.

At this point, we must recall also that the presence of non-Euclidean spatial topology implies the existence of distinct quantum sectors of any theory that includes gravity. Moreover, if we assume a fixed spatial topology, then these sectors do not communicate. Specifically, if we assume first that

$$^4M = \mathbb{R} \times M$$

(M being the topology of the spatial slices), and if we assume further that the meaningful assertions of the theory are all diffeomorphism invariant ("general covariance"), then we get *a distinct quantum sector for each UIR of the MCG of M*, where 'UIR' stands for 'unitary irreducible representation' and 'MCG' stands for 'mapping class group'. In fact, the mapping class group of a manifold M (also called "homeotopy group" or "group of

large diffeos") is the analog for gravity of the group of large gauge transformations in a gauge theory. Its formal definition is

$$G = \pi_0(\mathrm{Diff}^\infty(M)) := \mathrm{Diff}^\infty(M)/\mathrm{Diff}^\infty_0(M),$$

where $\mathrm{Diff}^\infty(M)$ is the group of diffeomorphisms of M that are trivial at infinity and $\mathrm{Diff}^\infty_0(M)$ is its connected subgroup. (An asymptotically flat approximation should be good for all but cosmological considerations.) In order to understand the different quantum sectors, we thus have to understand the structure of the homeotopy group G and then to use this information to analyze and interpret the different possible UIR's of G.

[The structure of the homeotopy group]

In this task a major help is that fact that G is generated * by only three types of diffeomorphism, each with a clear physical meaning. The three categories of generators are the *exchanges*, the *internal diffeomorphisms* and the *slides*. Like every diffeomorphism, each of these generators can be viewed as the result of a certain *process* (a "development" [2]), with the nature of the process being suggested by the name of the category. Thus, an exchange is the result of a process in which two identical primes continuously change places and a slide is the result of a process in which one prime travels around a loop threading through one or more other primes, while an internal diffeomorphism is a diffeomorphism whose support is restricted to a single prime. For example, to visualize an exchange of two handles in 2D, imagine the manifold as a rubber sheet, and imagine taking hold of the handles and dragging them around until they have changed places.

* This assertion is fully established in the mathematical literature for the orientable case, and appears to be true in the nonorientable case as well. More generally various statements we will make about the structure of G are in some cases known only under the assumption that the Poincaré conjecture is true, or that "homotopy implies isotopy" for the primes in question, or that the primes in question are "sufficiently large". All our statements are in any case known to be true for large families of primes, and are plausibly true in general. If they fail for certain primes, then the analysis given here will remain true as long as those primes are absent from the decomposition (1).

We spoke just now of "generators", but actually the slides and internals are already complete subgroups of G, while the exchanges generate a subgroup of G isomorphic to the group of all permutations of the identical (diffeomorphic) primes among themselves. The basic group theoretical fact we will need for our analysis is then that

$$G = (slides) \ltimes (internals) \ltimes (perms) \tag{2}$$

where the symbol $\ltimes$ denotes semidirect product (with the normal subgroup on the left). What this says more concretely is that every element of G is uniquely a product of three diffeomorphism-classes, one from each subgroup, and that each subgroup is invariant under conjugation by elements of the subgroups standing to its right in (2).

[The UIR's of a semidirect product group]

That fact that G is a semidirect product lets us analyze its UIR's in terms of representations of its factor groups and their subgroups. Indeed, there exists a very general analysis of the UIR's of a semidirect product, which is explained in detail in [1]. In essence it says the following. Let

$$G = N \ltimes K$$

be a semidirect product with N being the normal subgroup. A finite dimensional UIR of G is then determined by the following data

- $\Gamma =$ a UIR of N
- $T =$ a PUIR of $K_0 \subseteq K$,

where K_0 is the subgroup of K that remains "unbroken by Γ". (Often one calls K_0 "the little group".)

Perhaps the meaning of "unbroken" here can be illustrated most easily with the classic example of irreducible representations of the Poincaré group. There G is the Poincaré group itself, $N \simeq \mathbb{R}^4$ is its translation subgroup, and the quotient $K \simeq G/N$ is the Lorentz group. A choice of UIR Γ of $\mathbb{R}^4$ is then nothing but a choice of four-momentum P^μ, and (assuming that P^μ is timelike) the subgroup $K_0 \subseteq K$ left unbroken by this choice is $SO(3)$, the group of spatial rotations in the center of mass frame of

P^μ. The structure theorem for UIR's of semidirect products therefore tells us that we get a UIR of the Poincaré group (in this case an infinite dimensional UIR) by choosing a 4-momentum and a UIR T of $K_0 = SO(3)$. In particular language, we get a UIR by choosing two parameters, a mass * and a spin. In particular the "internal" properties of the particle (it's spin) are determined by the representation T of K_0.

For geons the formal situation is analogous, and the key interpretive point for us will be that the *statistics* of the geons will be determined by a representation of the "unbroken subgroup" of the group of permutations of identical primes.

Finally a comment on the "P" which occurs above in the phrase "PUIR of K_0." It stands for "projective", and reflects the fact that, even if one is seeking only ordinary representations of G, one may have to consider projective UIR's of K_0, i.e. representations up to a phase. Whether or not this occurs depends on the case in question; and when it does occur, the particular equivalence class of projective multipliers σ for K_0 which one must use is determined by the properties of the UIR Γ and how K_0 acts on it. (In the case of the Poincaré group, it does not occur, which is why we have to consider just integer spins, unless we want a spinorial representation of the overall group G itself.)

With these preparations complete, we can proceed to analyze the UIR's of the MCG of our spatial manifold M, and thereby the quantum sectors of gravity on $\mathbb{R} \times M$. One may distinguish two situations, according to whether the slide subgroup is represented trivially or not.

[The sectors with trivial slides]

In the simpler case of UIR's G which annihilate the slides, we may give in effect a complete classification. In this case, the mathematical problem is reduced to finding the UIR's of the quotient group, $G/(slides)$, which by (2) is just the semidirect product

$$(internals) \ltimes (perms). \tag{3}$$

* Only the mass counts, according to the theorem, because UIR's Γ of N which belong to the same K-orbit (in this case P^μ's lying on the same mass shell) yield equivalent representations of G.

Let us find the finite dimensional UIR's of this group (sometimes called the "particle group"), when only a single type of prime P is present in the decomposition (1). In this situation the permutation subgroup is just

$$K = (perms) = \mathcal{S}_N ,$$

the full permutation group on N elements (N being the number of copies of P which are present), and the group of internal diffeos is the direct product of N copies of the corresponding group $G^{(1)}$ for a single prime P:

$$N = (internals) = G^{(1)} \times G^{(1)} \times \cdots \times G^{(1)} .$$

From this last equation it follows immediately that the most general UIR of the normal subgroup N is itself a product, namely the tensor product

$$\Gamma = \Gamma_a \otimes \Gamma_a \cdots \Gamma_b \otimes \Gamma_b \cdots \Gamma_c . \tag{4}$$

Notice here that although all the underlying primes are all identical, there is no reason for all the factors in (4) to be so. Rather, we can choose an independent UIR of $G^{(1)}$ for each prime summand, and in the above formula, the subscripts a, b, $\ldots c$ label the different equivalence classes among them. Physically a given Γ_a specifies a certain "internal structure" for the corresponding geon, and is therefore a "species parameter" or "quantum number", analogous in many ways to the spin of a rigid nucleus or molecule in its body-centered frame.

With respect to the choice (4), it is intuitively clear (and true as well) that the unbroken subgroup $K_0 \subseteq (perms)$ reduces to a product of permutation groups,

$$K_0 = \mathcal{S}_{N_a} \times \mathcal{S}_{N_b} \times \cdots \times \mathcal{S}_{N_c} , \tag{5}$$

where N_a is the number of occurrences of the representation Γ_a, N_b of Γ_b, etc. The statistics is then given by a UIR T of K_0, that is to say by an independent UIR T_a, $T_b, \cdots$ for each of the subgroups $\mathcal{S}_{N_a}$, $\mathcal{S}_{N_b}, \cdots$. Each of these T's in turn, can be specified by a choice of a Young tableau, and determines whether the corresponding geons will manifest Bose statistics, Fermi statistics or some particular parastatistics. Since there is

no restriction on the choice of T, there is no restriction on which combinations of these possible statistics can occur.

Notice that these conclusions (deriving from the structure of K_0, as given in (5)) are entirely consistent with our interpretation of different choices of internal UIR in (4) as yielding physically distinct geons — geons of different "species". In this sense, we can say that there occurs a *quantum breaking of indistinguishability* conditioned by the choice of representation of ($internals$). This phenomenon, together with the possibility of assigning an arbitrary statistics to each such resulting species, exhausts the possibilities inherent in UIR's of the group (3). Thus, all possible sectors with trivial slides are accounted for by specifying

- a *species* for each geon (i.e. a UIR of $G^{(1)}$)
- a *statistics* for each resulting set of identical geons

[Some sectors with nontrivial slides]

When the slide subgroup is represented nontrivially, we are unable to give a full classification of the possible UIR's of G, due primarily to the difficulty of analyzing the UIR's of ($slides$), but also due in part to the relative complexity of the manner in which ($internals$) acts on these UIR's. Instead, let us consider a special case which avoids most of these complications, by choosing a prime that lacks internal diffeomorphisms, and then limiting ourselves mainly to abelian UIR's of the slides.

The prime in question is $\mathbb{R}P^3$, which can be visualized as a region of M produced by excising a solid ball and then identifying antipodal pairs of points on the resulting S^2 boundary. Since the internal group is trivial for this prime, we can concentrate on the effects of the slides. For each pair of $\mathbb{R}P^3$'s, one can slide one through the other, with the square of this slide being trivial (since $\pi_1(\mathbb{R}P^3) = \mathbf{Z}_2$), making a total of $N(N-1)$ independent order 2 generators. The complete group ($slides$) is then generated by products of these elementary slides, subject (when $N > 2$) to certain geometrically evident commutation relations, like the fact that slides involving disjoint subsets of the primes commute with each other. For abelian representations of ($slides$), all of the commutation relations will of course be satisfied trivially.

Since for $\mathbb{R}P^3$, $(internals)$ is trivial, the MCG reduces to

$$G = (slides) \ltimes (perms)$$

Hence, according to the general scheme outlined earlier, we get a UIR of G by choosing first a UIR Γ of $N = (slides)$, and then a PUIR T (with the correct projective multiplier σ) of the resulting unbroken subgroup $K_0 \subseteq (perms)$. As before, we may interpret K_0 as describing the surviving indistinguishability of the geons, and T as describing the statistics within each set of identical geons. Here we will just quote the results of this analysis, referring the listener to [1] for more details.

[A single pair of $\mathbb{R}P^3$*'s]*

This case is simple enough that we can classify all the UIR's of G, without limiting ourselves to abelian representations Γ. The quantum sectors comprise a 1-parameter continuous family together with a handful of discrete cases. Aside from the long-known violation of the spin-statistics correlation which occurs in one of the sectors, the intriguing new result is that there exist other sectors where the Bose-Fermi distinction becomes ambiguous in a certain sense. In these sectors the permutation group remains unbroken ($K_0 = \mathbf{Z}_2$), but there is no natural way to say which of its two representations describes a pair of bosons and which a pair of fermions! This happens because the UIR's Γ of $(slides)$ and T of $(perms)$ mix in such a way that it apparently becomes meaningless to identify either UIR of T as the trivial one.

[A trio of $\mathbb{R}P^3$*'s]*

Now we revert to the special case where Γ is abelian, meaning in effect that it merely associates a sign with each ordered pair of primes. We can represent the various such Γ pictorially by drawing three dots to represent the three primes and an arrow to represent each ordered pair that receives a minus sign (meaning the a slide of the first prime through the second produces a phase-factor of -1). Each distinct diagram of this type then gives rise to a different class of UIR's of $(slides)$, and therefore furnishes a different building block for constructing UIR's of G.

Perhaps the most interesting abelian UIR of ($slides$) comes from the cyclic graph in which the three dots and arrows form a circle. Clearly this pattern leaves $\mathbf{Z}_3 \subseteq (perms)$ as the unbroken subgroup K_0, so we acquire three distinct UIR's of G, corresponding to the three possible UIR's of $\mathbf{Z}_3$. What is remarkable here is first of all the pattern of geon identity, which is expressed not by a permutation group $\mathcal{S}_n$ at all, but by the cyclic group $\mathbf{Z}_3$. With this new type of group comes a new type of statistics, in which a cyclic permutation of the geons produces the complex phase q or $\overline{q}$, $q = 1^{1/3}$ being a cube-root of unity. *

Although this pattern of identity is unusual for the simple type of particle that physics usually deals with, it has obvious precedents in the social world of human beings. There it might happen, for example, that three people could stand in a triangle so that each was the teacher (in some different subject) of the one to his/her right. The pattern would then be preserved by a cyclic permutation, but not if two of the people changed places while the third stayed put.

[More than three $\mathbb{R}P^3$'s]

An interesting possibility in this case is that of "projective statistics", meaning a type of statistics expressed by a properly projective representation of the permutation group or one of its subgroups. † We do not have an example yet, but there seems to be no good reason why one shouldn't exist. We would need at least four geons because $\mathcal{S}_n$ possesses properly projective representations only for $n \geq 4$. We would also need a non-abelian UIR Γ of ($slides$), because in the contrary case, the projective multiplier σ will always be trivial. Thus, the simplest example one might try to construct, would employ a two dimensional representation Γ, so chosen that the unbroken subgroup $K_0 \subseteq \mathcal{S}_4$ would

* The same UIR of $\mathbf{Z}_3$ also occurs in connection with parastatistics, where it represents only a proper subspace of the full state-space. Indeed, any UIR of any finite group can be realized in connection with parastatistics, since any finite group is a subgroup of some permutation group.

† Unlike the "cyclic statistics" we just met, a projective statistics would not be realizable in connection with parastatistics.

come out as the subgroup $\mathcal{S}_4^{even}$ of even permutations, and σ would come out as the projective multiplier for the spin=1/2 representation of the symmetry group of a regular tetrahedron (to which $\mathcal{S}_4^{even}$ is isomorphic).

From the above examples it emerges clearly, we believe, that the possibility of non-Euclidean spacetime topology introduces a once unexpected richness into quantum gravity. In particular it brings with it

- topological particles (geons)
- half integer spin in pure gravity
- distinct quantum sectors (both continuous and discrete families of them)
- quantum multiplicity

We have not really discussed the second of these, and we did not mention the third at all before now, but we list them here to help illustrate the fertility of topology in the quantum context.

We have seen, moreover, that the mathematical structure theorems for representations of semidirect product groups provide a remarkably natural physical description of the different sectors which can occur. Indeed these theorems read almost as if they had been expressly designed to describe the representations in the language of quantum particles and their properties! In this language identical geon statistics is expressed by a (possibly projective) unitary irreducible representation of the unbroken subgroup of the group of permutations of identical primes. Two noteworthy features of the resulting interpretation are that

- All the familiar types of statistics can occur, together with some new, unexpected types.
- New patterns of particle identity occur, in which not all permutations of the identical particle leave the physics invariant.

It is interesting that the most novel of these features are associated with the slide diffeomorphisms, which correspond physically to processes in which one geon "slides thru"

another. For this reason it seems possible that analogous condensed matter effects could occur with objects like vortices, which also can slide through one another in an obvious manner [3].)

Beyond the existence of sectors associated with new types of particle statistics and indistinguishability, it seems that continuous parameters will also occur (we saw an example in the case of two $\mathbb{R}P^3$ geons), so that a great multiplicity of sectors can be expected to exist, even with a given spatial topology. This answers our initial question about new topological parameters, and it seems that the answer is that there are many — probably too many in fact, since they can be chosen so that the spin-statistics correlation fails, indicating * that quantum gravity is more akin to a phenomenological theory than a fundamental one. One can expect that unfreezing the topology will remove some of these unwanted sectors, but we would conjecture that a deeper, discrete theory will be needed to restore a physically reasonable degree of uniqueness to quantum gravity.

This research was partly supported by NSF grant PHY-9600620.

References

[1] R.D. Sorkin and S. Surya, "An Analysis of the Representations of the Mapping Class Group of a Multi-geon Three-manifold" ⟨e-print archive: gr-qc/9605050⟩

[2] R.D. Sorkin, "Introduction to Topological Geons", in P.G. Bergmann and V. de Sabbata (eds.), *Topological Properties and Global Structure of Space-Time*, pp. 249-270 (Plenum, 1986)

[3] A.P. Balachandran, private communication.

---

* If geons propagating in a flat ambient metric admit an approximate description in terms of an effective flat-space quantum field theory, then they must satisfy the standard spin-statistics correlation.